

COLLEGE ALGEBRA *and* CALCULUS

An Applied Approach

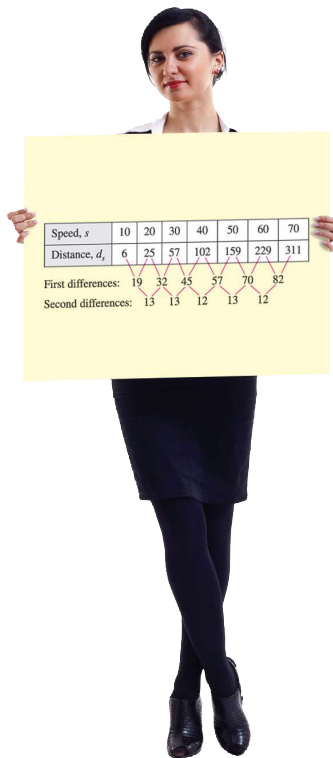


IT'S ALL ABOUT
YOU

Second Edition

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16.5 Mathematical Induction



In Exercise 76 on page 1192, you will use finite differences to determine whether the distances required to stop a truck at various speeds are better represented by a linear or a quadratic model.

- Use mathematical induction to prove a formula.
- Find a sum of powers of integers.
- Find a formula for a finite sum.
- Use finite differences to find a linear or quadratic model.

Introduction

In this section, you will study a form of mathematical proof called **mathematical induction**. It is important that you clearly see the logical need for it, so take a look at the problem below.

$$S_1 = 1 = 1^2$$

$$S_2 = 1 + 3 = 2^2$$

$$S_3 = 1 + 3 + 5 = 3^2$$

$$S_4 = 1 + 3 + 5 + 7 = 4^2$$

$$S_5 = 1 + 3 + 5 + 7 + 9 = 5^2$$

Judging from the pattern formed by these first five sums, it appears that the sum of the first n odd integers is

$$S_n = 1 + 3 + 5 + 7 + 9 + \cdots + (2n - 1) = n^2.$$

Although this particular formula *is* valid, it is important for you to see that recognizing a pattern and then simply *jumping to the conclusion* that the pattern must be true for all values of n is *not* a logically valid method of proof. There are many examples in which a pattern appears to be developing for small values of n and then at some point the pattern fails. One of the most famous cases of this was the conjecture by the French mathematician Pierre de Fermat (1601–1665), who speculated that all numbers of the form

$$F_n = 2^{2^n} + 1, \quad n = 0, 1, 2, \dots$$

are prime. For $n = 0, 1, 2, 3$, and 4 , the conjecture is true.

$$F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65,537$$

The size of the next Fermat number ($F_5 = 4,294,967,297$) is so great that it was difficult for Fermat to determine whether it was prime or not. However, another well-known mathematician, Leonhard Euler (1707–1783), later found the factorization

$$\begin{aligned} F_5 &= 4,294,967,297 \\ &= 641(6,700,417) \end{aligned}$$

which proved that F_5 is not prime and therefore Fermat's conjecture was false.

Just because a rule, pattern, or formula seems to work for several values of n , you cannot simply decide that it is valid for all values of n without going through a *legitimate proof*. Mathematical induction is one method of proof.

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The Principle of Mathematical Induction

Let P_n be a statement involving the positive integer n . If

1. P_1 is true, and
 2. the truth of P_k implies the truth of P_{k+1} for every positive integer k ,
- then P_n must be true for all positive integers n .

It is important to recognize that both parts of the Principle of Mathematical Induction are necessary. To apply the Principle of Mathematical Induction, you need to be able to determine the statement P_{k+1} for a given statement P_k .

Example 1 Using P_k to Find P_{k+1}

Find P_{k+1} for each statement.

a. $P_k: S_k = \frac{k^2(k+1)^2}{4}$

b. $P_k: S_k = 1 + 5 + 9 + \cdots + [4(k-1) - 3] + (4k-3)$

c. $P_k: 3^k \geq 2k + 1$

SOLUTION

$$\begin{aligned} \text{a. } P_{k+1}: S_{k+1} &= \frac{(k+1)^2(k+1+1)^2}{4} && \text{Replace } k \text{ by } k+1. \\ &= \frac{(k+1)^2(k+2)^2}{4} && \text{Simplify.} \end{aligned}$$

$$\begin{aligned} \text{b. } P_{k+1}: S_{k+1} &= 1 + 5 + 9 + \cdots + \{4[(k+1)-1] - 3\} + [4(k+1) - 3] \\ &= 1 + 5 + 9 + \cdots + (4k-3) + (4k+1) \end{aligned}$$

$$\begin{aligned} \text{c. } P_{k+1}: 3^{k+1} &\geq 2(k+1) + 1 \\ 3^{k+1} &\geq 2k + 3 \end{aligned}$$

✓ Checkpoint 1

Find P_{k+1} for $P_k: S_k = k(2k+1)$.



FIGURE 16.26

A well-known illustration used to explain why the Principle of Mathematical Induction works is the unending line of dominoes shown in Figure 16.26. If the line actually contains infinitely many dominoes, then it is clear that you could not knock the entire line down by knocking down only *one domino* at a time. However, suppose it were true that each domino would knock down the next one as it fell. Then you could knock them all down simply by pushing the first one and starting a chain reaction. Mathematical induction works in the same way. If the truth of P_k implies the truth of P_{k+1} and if P_1 is true, then the chain reaction proceeds as follows: P_1 implies P_2 , P_2 implies P_3 , P_3 implies P_4 , and so on.

Using Mathematical Induction

Example 2 Using Mathematical Induction

Use mathematical induction to prove the formula

$$S_n = 1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$$

for all integers $n \geq 1$.

SOLUTION Mathematical induction consists of two distinct parts. First, you must show that the formula is true when $n = 1$.

1. When $n = 1$, the formula is valid, because

$$S_1 = 1 = 1^2.$$

The second part of mathematical induction has two steps. The first step is to assume that the formula is valid for some integer k . The second step is to use this assumption to prove that the formula is valid for the next integer, $k + 1$.

2. Assuming that the formula

$$S_k = 1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2$$

is true, you must show that the formula $S_{k+1} = (k + 1)^2$ is true.

$$\begin{aligned} S_{k+1} &= 1 + 3 + 5 + 7 + \cdots + (2k - 1) + [2(k + 1) - 1] \\ &= [1 + 3 + 5 + 7 + \cdots + (2k - 1)] + (2k + 2 - 1) \\ &= S_k + (2k + 1) && \text{Group terms to form } S_k. \\ &= k^2 + 2k + 1 && \text{Replace } S_k \text{ by } k^2. \\ &= (k + 1)^2 && \text{Factor.} \end{aligned}$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for *all* integers $n \geq 1$.

STUDY TIP

When using mathematical induction to prove a *summation* formula (such as the one in Example 2), it is helpful to think of S_{k+1} as $S_{k+1} = S_k + a_{k+1}$, where a_{k+1} is the $(k + 1)$ th term of the original sum.

Checkpoint 2

Use mathematical induction to prove the formula

$$S_n = 4 + 5 + 6 + 7 + \cdots + (n + 3) = \frac{n(n + 7)}{2}$$

for all integers $n \geq 1$.

It occasionally happens that a statement involving natural numbers is not true for the first $k - 1$ positive integers but is true for all values of $n \geq k$. In these instances, you use a slight variation of the Principle of Mathematical Induction in which you verify P_k rather than P_1 . This variation is called the **Extended Principle of Mathematical Induction**. To see the validity of this variation, note from Figure 16.26 that all but the first $k - 1$ dominoes can be knocked down by knocking over the k th domino. This suggests that you can prove a statement P_n to be true for $n \geq k$ by showing that P_k is true and that P_k implies P_{k+1} . In Exercises 35–38 of this section, you are asked to apply this extension of mathematical induction.

Example 3 Using Mathematical Induction

Use mathematical induction to prove the formula

$$\begin{aligned} S_n &= 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 \\ &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

for all integers $n \geq 1$.

SOLUTION

1. When $n = 1$, the formula is valid, because

$$S_1 = 1^2 = \frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{1(2)(3)}{6}.$$

2. Assuming that

$$S_k = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

you must show that

$$S_{k+1} = \frac{(k+1)(k+1+1)[2(k+1)+1]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

To do this, write the following.

$$\begin{aligned} S_{k+1} &= S_k + a_{k+1} \\ &= (1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2) + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{By assumption} \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} && \text{Add expressions.} \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} && \text{Factor out } (k+1). \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} && \text{Simplify.} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} && \text{Completely factor.} \end{aligned}$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for *all* integers $n \geq 1$. _____

✓ Checkpoint 3

Use mathematical induction to prove the formula

$$S_n = 3 + 6 + 9 + 12 + \cdots + 3n = \frac{3n(n+1)}{2}$$

for all integers $n \neq 1$. ■

When proving a formula by mathematical induction, the only statement that you *need* to verify is P_1 . As a check, however, it is a good idea to try verifying some of the other statements. For instance, in Example 3, try verifying S_2 and S_3 .

Sums of Powers of Integers

The formula in Example 3 is one of a collection of useful summation formulas. This and other formulas dealing with the sums of various powers of the first n positive integers are shown.

Sums of Powers of Integers

1. $1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}$
2. $1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
3. $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$
4. $1^4 + 2^4 + 3^4 + 4^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$
5. $1^5 + 2^5 + 3^5 + 4^5 + \cdots + n^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

Each of these formulas for sums can be proven by mathematical induction.

Example 4 Finding a Sum of Powers of Integers

Find $\sum_{n=1}^7 n^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3$.

SOLUTION Using the formula for the sum of the cubes of the first n positive integers, you obtain the following.

$$\begin{aligned}
 \sum_{n=1}^7 n^3 &= 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 \\
 &= \frac{7^2(7+1)^2}{4} \\
 &= \frac{49(64)}{4} \\
 &= 784
 \end{aligned}$$

Check this sum by adding the numbers, as shown below.

$$1 + 8 + 27 + 64 + 125 + 216 + 343 = 784$$

✓ Checkpoint 4

Find $\sum_{n=1}^{10} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2$. ■

Example 5 Proving an Inequality by Mathematical Induction

Prove that $n < 2^n$ for all positive integers n .

SOLUTION

1. For $n = 1$ and $n = 2$, the formula is true, because

$$1 < 2^1 \text{ and } 2 < 2^2.$$

2. Assuming that

$$k < 2^k$$

you need to show that $k + 1 < 2^{k+1}$. For $n = k$, you have

$$2^{k+1} = 2(2^k) > 2(k) = 2k. \quad \text{By assumption}$$

Because $2k = k + k > k + 1$ for all $k > 1$, it follows that

$$2^{k+1} > 2k > k + 1$$

or

$$k + 1 < 2^{k+1}.$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that $n < 2^n$ for all positive integers n .

 Checkpoint 5

Prove that $n! > 2n$ for all integers $n > 3$.

Pattern Recognition

Although choosing a formula on the basis of a few observations does *not* guarantee the validity of the formula, pattern recognition *is* important. Once you have a pattern that you think works, you can try using mathematical induction to prove your formula.

Finding a Formula for the n th Term of a Sequence

To find a formula for the n th term of a sequence, consider these guidelines.

1. Calculate the first several terms of the sequence. It is often a good idea to write the terms in both simplified and factored forms.
2. Try to find a recognizable pattern for the terms and write a formula for the n th term of the sequence. This is your *hypothesis* or *conjecture*. You might try computing one or two more terms in the sequence to test your hypothesis.
3. Use mathematical induction to prove your hypothesis.

Example 6 Finding a Formula for a Finite Sum

Find a formula for the finite sum.

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{n(n+1)}$$

SOLUTION Begin by writing out the first few sums.

$$S_1 = \frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{1}{1+1}$$

$$S_2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{4}{6} = \frac{2}{3} = \frac{2}{2+1}$$

$$S_3 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{9}{12} = \frac{3}{4} = \frac{3}{3+1}$$

$$S_4 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} = \frac{48}{60} = \frac{4}{5} = \frac{4}{4+1}$$

From this sequence, it appears that the formula for the k th sum is

$$\begin{aligned} S_k &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{k(k+1)} \\ &= \frac{k}{k+1}. \end{aligned}$$

To prove the validity of this hypothesis, use mathematical induction, as shown below. Note that you have already verified the formula for $n = 1$, so you can begin by assuming that the formula is valid for $n = k$ and trying to show that it is valid for $n = k + 1$.

$$\begin{aligned} S_{k+1} &= \left[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{k(k+1)} \right] + \frac{1}{(k+1)(k+2)} \\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} && \text{By assumption} \\ &= \frac{k(k+2) + 1}{(k+1)(k+2)} && \text{Add fractions.} \\ &= \frac{k^2 + 2k + 1}{(k+1)(k+2)} && \text{Distributive Property} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} && \text{Factor numerator.} \\ &= \frac{k+1}{k+2} && \text{Simplify.} \end{aligned}$$

So, the hypothesis is valid. _____

✓ Checkpoint 6

Find a formula for the finite sum.

$$3 + 5 + 7 + 9 + \cdots + (2n+1)$$

